Simultaneous containment of several polygons : analysis of the contact configurations

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Simultaneous Containment of Several Polygons : Analysis of the contact configurations

Placement de plusieurs polygones : analyse des configurations de contacts*

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Abstract

The main concern of this paper is the detection of double-contact configurations for some polygons moving in translation in a polygonal environment. We first establish some general properties about such configurations, and give conditions of existence of double-contacts for two or three objects. We deduce an algorithm detecting such a position in time $O(n^2)$ (*resp.* $O(n^3)$) for two (*resp.* three) convex polygons moving in a non convex polygon.

Résumé

Dans cet article, nous cherchons à détecter des configurations de double-contacts pour plusieurs polygones en translation dans un environnement polygonal. Après léenoncé de quelques propriétés générales dans le cas de plusieurs objets, nous nous intéressons plus particulièrement aux cas de deux ou trois objets, et examinons les conditions permettant d'établir l'existence de double-contacts. On en déduit un algorithme en $O(n^2)$ (*resp.* $O(n^3)$) pour détecter un tel placement dans le cas de deux (*resp.* trois) polygones convexes en translation dans un environnement polygonal quelconque.

Introduction

Problems of placement polygons containment may be seen from two points of view : the search of one solution or the search of all the solutions (the free configurations space). For two convex polygons in a convex environment, the free space could be found in linear time [GRS83] ; in the case of general polygons in general polygonal environment Avnaim and Boissonnat [AB87,Avn89] gave algorithms to compute the free space or to only find one solution for two or three polygons. The complexity depends on the different cases of convexity of objects and environment, particularly they found one solutions for the containment of two (*resp.* three) convex polygons in a general polygons in $O(n^2 \log^2 n)$ (*resp.* $O(n^3 \log^2 n)$) if n is the environment size.

This paper presents some results about contact configurations for translating polygons in the plane. We just take interest in contact configurations and not in the entire free space. It is easy to show that for two polygons, if the free-space is non empty, a configuration of *double-contact* exists. This paper proves this property for three convex polygons in a polygonal environment or for three polygons moving in a rectangle. An algorithm of complexity $O(n^2)$ (*resp.* $O(n^3)$) is deduced to place two (*resp.* three) convex polygons. Examples are given to show that this property is false in other cases (three non convex polygons in a non convex environment or four convex polygons in a rectangle).

1 Definitions and generalities

1.1 Notations and classical results

We take interest in this paper about some polygonal objects moving in translation in a polygonal environment. The closure of the environment will be denoted by $\mathcal{O}$ and the closure of the objects by $\mathcal{S}_i$ $(i = 1 \dots q)$, n (*resp.* m_i) is the size of $\mathcal{O}$ (*resp.* $\mathcal{S}_i$). The position of each object is characterized by the translation of the plane $c_i \in I\!R^2$ which moves $\mathcal{S}_i$ from its reference position to its current position. c_i is called a configuration of $\mathcal{S}_i$ and we note $\mathcal{S}_i^{c_i}$ the object $\mathcal{S}_i$ in the configuration c_i. An usual way to represent a position for a system of q objects is to use a $2q$-dimensional space : the configuration space. $c = (c_1, \dots, c_q) \in I\!R^{2q}$ is a configuration of the complete system.

We now define a free configuration as a configuration where the closure of the objects have no intersection, and a contact configuration as a configuration where the interior of the objects have no intersection but their boundary have ones. As Avnaim [Avn89] let us denote :

$$
\begin{aligned}
\mathcal{L} &= \{c = (c_1, \dots, c_q), \forall i\ \mathcal{S}_i^{c_i} \cap \mathcal{O} = \emptyset \text{ and } \forall i, j\ i \neq j, \mathcal{S}_i^{c_i} \cap \mathcal{S}_j^{c_j} = \emptyset\} \\
&\quad \text{is the free space} \\
\mathcal{C} &= \{c = (c_1, \dots, c_q), c \notin \mathcal{L}, \forall i\ \breve{\mathcal{S}}_i^{c_i} \cap \breve{\mathcal{O}} = \emptyset \text{ and } \forall i, j\ i \neq j, \breve{\mathcal{S}}_i^{c_i} \cap \breve{\mathcal{S}}_j^{c_j} = \emptyset\} \\
&\quad \text{is the contact space } (\breve{\mathcal{A}} \text{ denote the interior of a set } \mathcal{A}) \\
\mathcal{L}_i &= \{c_i, \mathcal{S}_i^{c_i} \cap \mathcal{O} = \emptyset\} \\
&\quad \text{is the free space of only one object } \mathcal{S}_i \text{ in the environment } \mathcal{O} \\
\mathcal{L}_{ij} &= \{c_i - c_j, \mathcal{S}_i^{c_i} \cap \mathcal{S}_j^{c_j} = \emptyset\} \\
&\quad \text{is the free space of } \mathcal{S}_i \text{ relatively to } \mathcal{S}_j \\
\mathcal{U}_{ij} &= \{c_i - c_j, c_i \in \mathcal{L}_i, c_j \in \mathcal{L}_j, c_i - c_j \in \mathcal{L}_{ij}\} \\
&\quad \text{is the set of relative free configurations of } \mathcal{S}_i \text{ and } \mathcal{S}_j \text{ in } \mathcal{O} \\
\mathcal{R}_{ij} &= \{c_i - c_j, \exists c \in \mathcal{L},\ c_i \text{ and } c_j \text{ are the } i^{\text{th}} \text{ and } j^{\text{th}} \text{ components of } c\} \\
&\quad \text{is the set of relative free configurations of } \mathcal{S}_i \text{ and } \mathcal{S}_j \text{ in the complete} \\
&\quad \text{system of } q \text{ objects in the environment } \mathcal{O}
\end{aligned}
$$

We suppose that the system is in a *general disposition.* Degenerate cases as constraint motion in a corridor, and parallelism between objects are rejected (see Figure

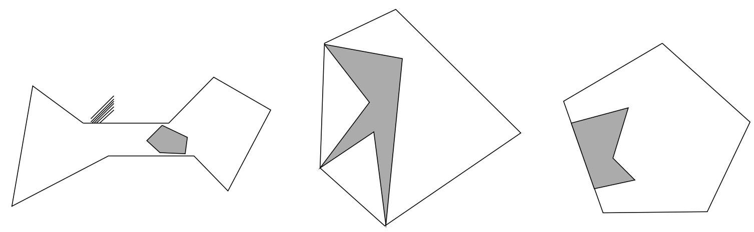

Figure 1: Some degenerate cases

1). An exact definition will be given in Section 2.1. If the system does not verify this hypothesis, a small deformation of objects or environment makes it in a general disposition.

For a polygonal system all these sets are (non necessarily convex) polytopes. $\mathcal{L}_i$, $\mathcal{L}_{ij}, \mathcal{U}_{ij}$ and $\mathcal{R}_{ij}$ are open polygons. $\mathcal{L}$ is an open polytope of dimension $2q$ and $\mathcal{C}$ is its boundary.

An usual way to compute $\mathcal{L}_i$ and $\mathcal{L}_{ij}$ is to use the Minkowski difference (denoted by $\ominus$) see for example [LW79] ($\bar{A}$ is the complementary set of A).

$$\bar{\mathcal{L}_i} = \mathcal{O} \ominus \mathcal{S}_i \tag{1}$$
$$\bar{\mathcal{L}_{ij}} = \mathcal{S}_j \ominus \mathcal{S}_i \tag{2}$$

The Minkowski difference of two polygons of size n and m can be computed in time $O(n^2m^2)$ and has a size $O(n^2m^2)$, this is tight in the worst case. To do this, one can use a general algorithm to compute the arrangement of the $O(nm)$ lines representing simple contacts [Ede87] and to choose which cells belong to the Minkowski difference and which does not. For a general polygon and a convex polygon, the result can be computed in time $O(nm \log nm)$ and has a size $O(nm)$ [For85]. In the case of two convex polygons we can achieve the computation in time $O(n+m)$ and the size is also $O(n+m)$ [GRS83].

Avnaim shows some results :

$$\mathcal{U}_{ij} = \mathcal{L}_{ij} \cap (\mathcal{L}_i \ominus \mathcal{L}_j) \tag{3}$$
$$\mathcal{R}_{ij} \subset \mathcal{U}_{ij} \cap (\mathcal{U}_{ik} \ominus \mathcal{U}_{jk}) \tag{4}$$
$$\text{where } i \neq j \neq k \neq i$$

and in the special case of three objects in a rectangle :

$$\mathcal{R}_{ij} = \mathcal{U}_{ij} \cap (\mathcal{U}_{ik} \ominus \mathcal{U}_{jk}) \quad \text{where } i,j,k \in \{1,2,3\}, i \neq j \neq k \neq i \tag{5}$$

1.2 Aims

This paper establishes some results about contact configurations for polygons moving in translation in the plane.

Part 2 summarizes some considerations about the contact space and establishes a necessary and sufficient condition of concavity for a vertex of $\mathcal{L}$, in Part 3 the existence of a double-contact is proved in two particular cases for three objects : three convex polygons in a polygonal environment or three polygons in a rectangle. This property is false for more than three polygons or for three non convex polygons in a non convex environment. Finally Part 4 deduces from the existence of double contacts, an algorithm to find one contact configuration if it exists one in time[1] $O(n^2m^2\log m)$ (*resp.* $O(n^3m^3\log m)$) for two (*resp.* three) convex polygons. Avnaim and Boissonnat's algorithms run in this case with a time complexity of $O(n^3m^2\log^2 nm)$ (*resp.* $O(n^3m^3\log^2 nm+n^3m^5\log m)$).

2 The contact space

As we noticed in Section 1.1 the configuration space has dimension $2q$ and the free-space is a polytope in this space.

2.1 Contact configuration

We call a *contact* a configuration of $\mathcal{C}$ as defined in Section 1.1. In a contact $c = (c_1, \ldots, c_q)$ there exist two objects $\mathcal{S}_i$ and $\mathcal{S}_j$ (or an object and the environment) such that $\mathcal{S}_i^{c_i}$ and $\mathcal{S}_j^{c_j}$ have a common point on their boundary, this point corresponds to a vertex V of $\mathcal{S}_i$ and to an edge e of $\mathcal{S}_j$ without loss of generality. Such a couple (V, e) is called *label of simple contact.*

Now let (V, e) be a label of simple contact. To impose that the translation c_i of $\mathcal{S}_i$ leads the vertex V of $\mathcal{S}_i$ on the supporting line of the edge e of $\mathcal{S}_j$ translated by c_j, the configuration c describes an hyperplane in the configuration space. More precisely, we have a linear constraint on the two dimensional vector $c_i - c_j$. The contact-space $\mathcal{C}$ is supported by such hyperplanes called *contact hyperplanes.*

We can now define what a system in *general diposition* means.

Definition 2.1 *The system is said in* general disposition *if the arrangement of all contact hyperplanes in the* $2q$*-dimensional space is in general position.*

(i.e. the intersection of $k \leq 2q$ *hyperplanes is a subspace of dimension* $2q - k$*, the intersection of* $k > 2q$ *hyperplanes is empty.)*

As claimed in Section 1.1, a small perturbation of objects can turn any system in general disposition. In this case $\mathcal{C}$ is the boundary of $\mathcal{L}$.

Let us introduce some vocabulary, we call a *simple contact*, a point of $\mathcal{C}$ lying on a contact hyperplane, a k*-contact* is a point of $\mathcal{C}$ lying on k contact hyperplanes, a k*-double contact* is a k-contact such that two hyperplanes have labels **involving the same pair of objects** otherwise it is a k*-simple contact.*

Let c be a point on the hyperplane of label (V, e), V is between the two end-points E and E' of e if c lies in a *validity slab* traced on the hyperplane. In other words, the slab

[1] for convenience we suppose that $m_i = O(m)$ and that $m \leq n$. For a more precise expression of complexity, see Section 4.

corresponds to a contact between V and e and not only the line supporting e. A point in this slab is a contact configuration with label (V, e). A point on the boundary of this slab is a contact vertex-vertex between the two objects with label (V, E) or (V, E') (a contact vertex-vertex (V, E) corresponds to two labels of simple contacts : (V, e) and (V, e') where E is the common end-point of edges e and e', this a 2-double contact[2]).

Usual properties of arrangements and the general disposition hypothesis yield the following proposition :

Proposition 2.1 *$\mathcal{C}$ is a polytope of dimension $2q$, the $(2q-k)$-faces of $\mathcal{C}$ correspond to k-contacts.*

2.2 Vertices of $\mathcal{C}$

This section examines under which conditions a vertex of $\mathcal{C}$ (or $\mathcal{L}$, $\mathcal{C}$ is the boundary of $\mathcal{L}$) may be concave or convex[3].

Let us focus on contact hyperplanes. $\mathcal{L}$ is a union of cells of the arrangement of contact hyperplanes. Let now c be a point on a contact hyperplane H, if c is strictly outside the validity slab then H cannot be a face of $\mathcal{C}$ in a neighborood of c. If c is strictly inside the validity slab then in a neighborood of c, $\mathcal{L}$ must be only on one side of H, so in this neighborood c is included in one of the two half spaces bounded by H (Figure 2).

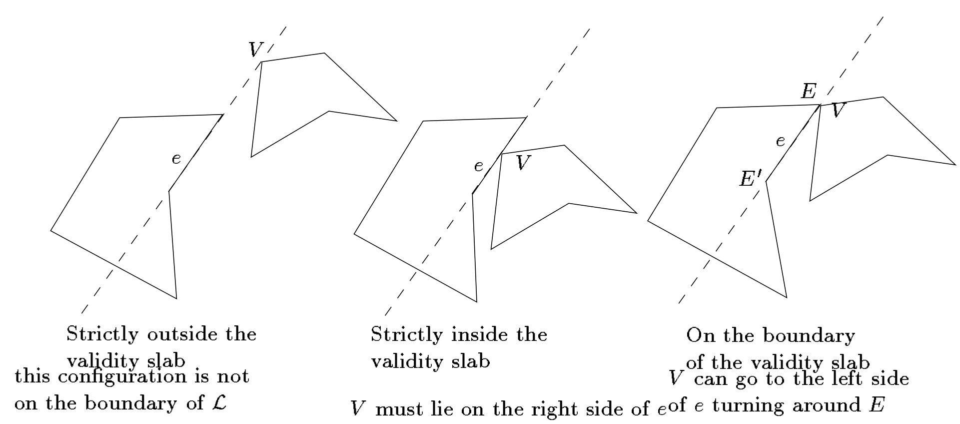

Figure 2: Configuration in the contact hyperplane

So let now c be a vertex of $\mathcal{C}$, Proposition 2.1 proves that c is a $2q$-contact configuration. It is the intersection of $2q$ contact hyperplanes defining the faces of $\mathcal{C}$ adjacent

[2]These two double-contacts (V, E) and (V, E') define two subspace of $\mathbb{R}^{2q}$ of dimension $2q-2$ and having the same direction. They are two parallels *hyper-hyperplanes* of the hyperplane corresponding to the contact (V, e).

[3]We define a concave vertex as a non convex vertex. A vertex is convex, if in a neighbourhood the polytope is an intersection of half spaces.

to c. As said above c cannot be strictly outside the validity slabs of the plane, so if c is strictly in all these slabs then locally $\mathcal{L}$ is the intersection of $2q$ half spaces and the vertex is convex.

Otherwise, c is on the boundary of a validity slab, c corresponds to a double contact vertex vertex (V, E) between two objects $\mathcal{S}_i$ and $\mathcal{S}_j$ (one may be the environment $\mathcal{O}$, see Figure 3). Now, we only have a two dimensional problem and it is easy to see that a valid double contact vertex-vertex (V, E) corresponds to a concave vertex in $\mathcal{C}$ if and only if the two vertices V and E are both convex.

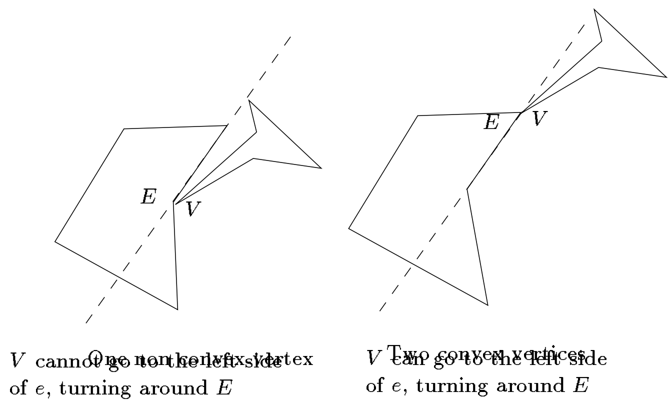

Figure 3: Reduction to 2D problem

Proposition 2.2 *A vertex c of $\mathcal{C}$ is concave if and only if it corresponds to a $2q$-contact with a double-contact between two convex vertices of two objects (or one convex vertex of an object and one convex vertex of the environment).*

3 Double-contacts

$\mathcal{C}$ is a polytope whose vertices correspond to $2q$-contact configurations. Each of these vertices may be a $2q$-simple contact or a double contact. We take interest in detecting the existence of double-contacts. In other words we want to solve the following question : *does there exist a connected subset of $\mathcal{C}$ whose vertices are all $2q$-simple contacts ?*

3.1 Two polygons

In the case of only two objects, a vertex of $\mathcal{C}$ is a 4-contact, but there are only three different possibilities of contacts ($\mathcal{O} - \mathcal{S}_1$, $\mathcal{O} - \mathcal{S}_2$, $\mathcal{S}_1 - \mathcal{S}_2$). So a 4-contact is necessarily a double-contact.

3.2 Three polygons

From a combinatorial point of view, the case of three objects is particular. A vertex of $\mathcal{C}$ is a 6-contact and there are exactly six different possibilities of contacts : $\mathcal{O} - \mathcal{S}_1$, $\mathcal{O} - \mathcal{S}_2$, $\mathcal{O} - \mathcal{S}_3$, $\mathcal{S}_1 - \mathcal{S}_2$, $\mathcal{S}_1 - \mathcal{S}_3$, $\mathcal{S}_2 - \mathcal{S}_3$. This fact allows us to determine special properties in the case of three objects.

If we suppose that a connected component of $\mathcal{C}$ does not contain double-contact configurations ; a vertex c of this component is a 6-simple contact.

Let $(l_{\mathcal{O}\mathcal{S}_1}, l_{\mathcal{O}\mathcal{S}_2}, l_{\mathcal{O}\mathcal{S}_3}, l_{\mathcal{S}_1\mathcal{S}_2}, l_{\mathcal{S}_1\mathcal{S}_3}, l_{\mathcal{S}_2\mathcal{S}_3})$ be the label describing this contact. ($l_{\mathcal{F}\mathcal{G}}$ is a label of simple contact between $\mathcal{F}$ and $\mathcal{G}$.)

Among the six edges having c as an extremity (that corresponds to a 5-contact using Proposition 2.1) we choose the edge e having $(l_{\mathcal{O}\mathcal{S}_1}, l_{\mathcal{O}\mathcal{S}_2}, l_{\mathcal{O}\mathcal{S}_3}, l_{\mathcal{S}_1\mathcal{S}_3}, l_{\mathcal{S}_2\mathcal{S}_3})$ as label of 5-simple contact. When the configuration moves along this edge, the contact between $\mathcal{S}_1$ and $\mathcal{S}_2$ disappears.

Let c' be the other end-point of e. To agree with our hypothesis, c' is a 6-simple contact different to c, its label is $(l_{\mathcal{O}\mathcal{S}_1}, l_{\mathcal{O}\mathcal{S}_2}, l_{\mathcal{O}\mathcal{S}_3}, l'_{\mathcal{S}_1\mathcal{S}_2}, l_{\mathcal{S}_1\mathcal{S}_3}, l_{\mathcal{S}_2\mathcal{S}_3})$ with $l_{\mathcal{S}_1\mathcal{S}_2} \neq l'_{\mathcal{S}_1\mathcal{S}_2}$. From c' it is possible to get the edge e' having label $(l_{\mathcal{O}\mathcal{S}_1}, l_{\mathcal{O}\mathcal{S}_2}, l_{\mathcal{O}\mathcal{S}_3}, l'_{\mathcal{S}_1\mathcal{S}_2}, l_{\mathcal{S}_2\mathcal{S}_3})$. The other extremity of e' is c'' having label $(l_{\mathcal{O}\mathcal{S}_1}, l_{\mathcal{O}\mathcal{S}_2}, l_{\mathcal{O}\mathcal{S}_3}, l'_{\mathcal{S}_1\mathcal{S}_2}, l''_{\mathcal{S}_1\mathcal{S}_3}, l_{\mathcal{S}_2\mathcal{S}_3})$ with $l_{\mathcal{S}_1\mathcal{S}_3} \neq l''_{\mathcal{S}_1\mathcal{S}_3}$.

In this manner, it is possible to turn around the 2-face of the connected component of $\mathcal{C}$ corresponding to the 4-simple contact of label $(l_{\mathcal{O}\mathcal{S}_1}, l_{\mathcal{O}\mathcal{S}_2}, l_{\mathcal{O}\mathcal{S}_3}, l_{\mathcal{S}_2\mathcal{S}_3})$. To alternate on each edge the missing contact $(\mathcal{S}_1 - \mathcal{S}_2)$ or $(\mathcal{S}_1 - \mathcal{S}_3)$ this 2-face must have an even number of vertices and at least be a quadrilateral.

The following properties are easily deduced :

· a 2-face is at least a quadrilateral.

· a 3-face is at least a parallelepiped.

⋮

· a 6-face (i.e. a connected component) is at least a hyper-parallelepiped.

Moreover Proposition 2.2 assures that a 6-simple contact is a convex vertex of $\mathcal{C}$. The following proposition summarizes these results about the shape and the size of such a connected component of $\mathcal{C}$:

Proposition 3.1 *In a system of three polygons in a polygonal environment, a connected component of $\mathcal{C}$ which does not contain double-contact has at least 64 vertices and is convex.*

Such a system exists, see Figure 4. In this example, there are exactly twelve labels of simple contact (two for each possibility of contacts, these simple contacts are marked by arrows on Figure 4), there are 64 ways to choose a 6-simple contact label combining these twelve labels, each of these possibilities corresponds to a vertex of $\mathcal{C}$.

3.3 Three convex polygons

In some particular cases, the existence of a double-contact can be proved.

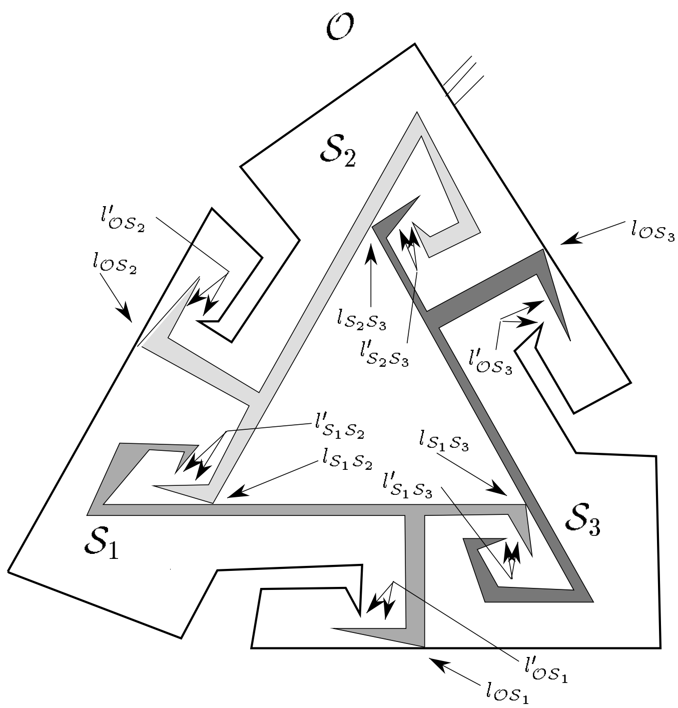

Figure 4: Free space without double-contact

Let $\mathcal{S}_1$,$\mathcal{S}_2$ and $\mathcal{S}_3$ be three convex polygons and $\mathcal{O}$ a polygonal environment. Let c be a 6-simple contact configuration, vertex of $\mathcal{C}$. From c, breaking the contact $\mathcal{S}_1 - \mathcal{S}_2$, it is possible to move along the corresponding edge. This movement in the configuration space corresponds to a relative translation moving $\mathcal{S}_1$ away from $\mathcal{S}_2$. Because $\mathcal{S}_1$ and $\mathcal{S}_2$ are both convex, it is impossible to restore the contact $\mathcal{S}_1 - \mathcal{S}_2$ at the other extremity of the edge (even with another label), it is necessary to introduce another contact already existing to restore a 6-contact at the end of the edge. So a double-contact configuration exists.

It is possible to claim a stronger result. From c a 6-contact, one can move on the 3-face corresponding to three contacts $\mathcal{S}_i - \mathcal{O}$ in the configuration space. Moving the configuration point along an half-line drawn on this 3-face corresponds to a regular movement of the three objects moving each convex polygon away from the two others. The intersection between the half-line and the 3-face is a segment which second extremity is at least a 4-contact (at most a 2-face). As objects are moving away, the new contact is necessarily a contact between a convex polygon and the environment, and so a double-contact. Unfortunately, it is not always possible to find another double-contact and Figure 5 shows an example where $\mathcal{S}_1$ is the only object able to got a double-contact with $\mathcal{O}$. The little pictures represent the other vertices of $\mathcal{C}$.

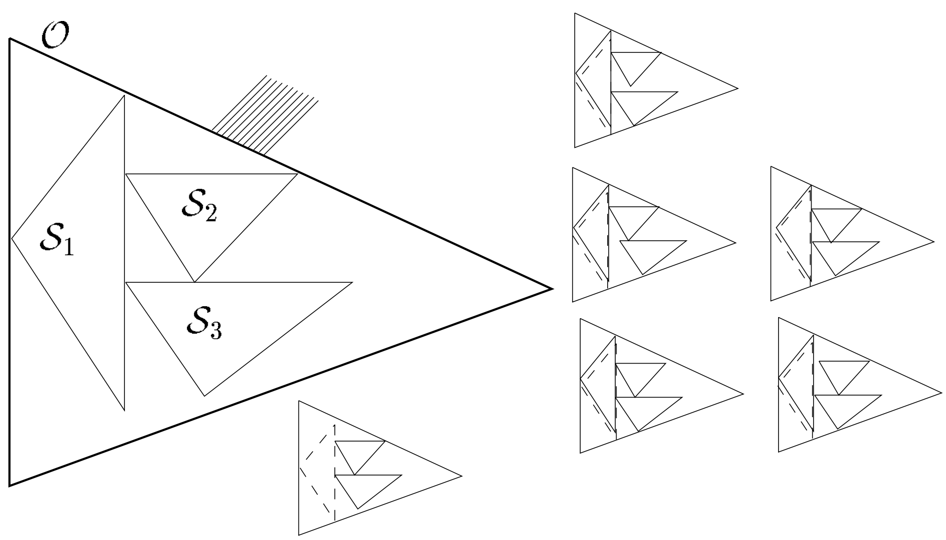

Figure 5: Three convex polygons with only one possibility of double-contact

Proposition 3.2 *In a system of three convex polygons in a polygonal environment there exists a configuration with a double contact between a polygon and the environment in each connected component of $\mathcal{L}$.*

In the same way it is possible to prove that :

Proposition 3.3 *In a system of two convex polygons in a polygonal environment there exists a configuration where both polygons have a double contact with the environment in each connected component of $\mathcal{L}$.*

3.4 Three polygons in a rectangular environment

First, we prove an easy geometric lemma :

Lemma 3.1 *If A, B and C are three polygons in general disposition.*
If r is a vertex of $A \cap (B \ominus C)$ then
(r is a vertex of A) or
($[b \in B, c \in C, r = b - c] \Rightarrow b$ is a vertex of B or c is a vertex of C).

Proof : r is a vertex of A, r is a vertex of $B \ominus C$ or r is the intersection of an edge of A and an edge of $B \ominus C$.

It is well known that a vertex of $B \ominus C$ is a difference of vertices of B and C and that an edge of $B \ominus C$ is a difference of a vertex of B and an edge of C or an edge of B and a vertex of C.

So one can conclude. □

The following proposition arises now :

Proposition 3.4 *In a system of three polygons in a rectangular environment there exists a configuration with a double contact (if $\mathcal{L} \neq \emptyset$).*

Proof : If $\mathcal{C}$ is non empty, $\mathcal{R}_{12}$ is also non empty and has a vertex c_{12}. Using lemma 3.1 and equation 5, c_{12} is a vertex of $\mathcal{U}_{12}$ or it exists c_{13} and c_{23} with $c_{12} = c_{13} - c_{23}$ and c_{13} is a vertex of $\mathcal{U}_{13}$ or c_{23} is a vertex of $\mathcal{U}_{23}$. So we have : *it exists c_{ij} vertex of $\mathcal{U}_{ij}$ corresponding to a free configuration in the complete system.*

Applying again lemma 3.1 and equation 3 we deduce that c_{ij} is a vertex of $\mathcal{L}_{ij}$ (so there exists a double contact between $\mathcal{S}_i$ and $\mathcal{S}_j$), or $c_{ij} = c_i - c_j$ and c_k ($k = i$ or j) is a vertex of $\mathcal{L}_k$ (so there exists a double contact between $\mathcal{S}_k$ and $\mathcal{O}$). □

3.5 More than three objects

As it is shown in Figure 6 it is possible to have a connected component of the free space without double-contact, even for four convex polygons in a rectangular environment. In this example the free-space is a simplex, 9 labels of simple contact are marked on the figure and the 9 vertices of $\mathcal{C}$ are defined by choosing 8 labels among these 9.

4 Placing convex polygons

An algorithm to detect a free configuration (if there exists one) for two or three convex polygons in any polygonal environment can be deduced from the preceding section.

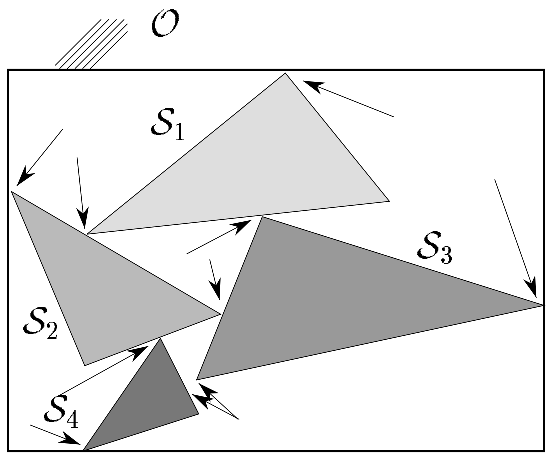

Figure 6: Four objects without double-contact

4.1 Two convex polygons

Proposition 3.3 assures that if $\mathcal{L}$ is non empty, there exists a 4-contact configuration where both objects have a double-contact with the environment. To detect such a placement, we can use the following algorithm :

1. Compute $\mathcal{L}_1$ using Minkowski difference.
2. Compute $\mathcal{L}_2$.
3. Compute $\mathcal{L}_{12}$.
4. For each pair of vertices $(c_1, c_2) \in \mathcal{L}_1 \times \mathcal{L}_2$ test if $c_1 - c_2 \in \mathcal{L}_{12}$.

We can study the complexity of this algorithm. n (*resp.* m_1,m_2) is the size of $\mathcal{O}$ (*resp.* $\mathcal{S}_1$, $\mathcal{S}_2$).

- Step 1 consists in computing the Minkowski difference of a convex polygon and a general polygon. It can be done in time $O(nm_1 \log nm_1)$ and the size of $\mathcal{L}_1$ is $O(nm_1)$.
- In the same way step 2 can be completed in time $O(nm_2 \log nm_2)$ and the size of $\mathcal{L}_2$ is $O(nm_2)$.
- In step 3, both polygons are convex, the Minkowski difference can be computed in time $O(m_1 + m_2)$ and $\mathcal{L}_{12}$ is a convex polygon of size $O(m_1 + m_2)$.
- In fact, the limiting step is step 4. The number of cases to examine is $O(n^2 m_1 m_2)$. For each case we have to test if a point is inside or outside a convex polygon of size $O(m_1 + m_2)$ in time $O(\log(m_1 + m_2)) = O(\log m_1 m_2)$.

The time complexity of this algorithm is $O(n^2 m_1 m_2 \log m_1 m_2)$ which improves over the result of [AB87] ($O(n^2 m_1 m_2 \log nm_1 \log nm_2)$).

4.2 Three convex polygons

From the proposition 3.2 if $\mathcal{L}$ is non empty, there exists a 6-contact configuration with a double contact between one object $\mathcal{S}_i$ and $\mathcal{O}$. The following algorithm searches such a configuration :

1. Compute $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.
2. For each vertex c_1 of $\mathcal{L}_1$, search if the system with two objects $\mathcal{S}_2$ and $\mathcal{S}_3$ in the environment $\mathcal{O} \cup \mathcal{S}_1^{c1}$ has a free configuration using the preceding algorithm.
3. For each vertex c_2 of $\mathcal{L}_2$, search if the system with two objects $\mathcal{S}_1$ and $\mathcal{S}_3$ in the environment $\mathcal{O} \cup \mathcal{S}_2^{c2}$ has a free configuration.
4. For each vertex c_3 of $\mathcal{L}_3$, search if the system with two objects $\mathcal{S}_1$ and $\mathcal{S}_2$ in the environment $\mathcal{O} \cup \mathcal{S}_3^{c3}$ has a free configuration.

The complexity of this algorithm is studied below :

- Step 1 is done in time $O(\sum_i nm_i \log nm_i)$ and the size of $\mathcal{L}_i$ is $O(nm_i)$.
- In step 2, we have to examine $O(nm_1)$ cases. Each case is solved in time : $O((n+m_1)^2 m_2 m_3 \log m_2 m_3)$ (see section 4.1).
- Steps 3 and 4 are identical to the preceding one. Their time complexity are : $O(nm_2(n+m_2)^2 m_1 m_3 \log m_1 m_3)$ and $O(nm_3(n+m_3)^2 m_1 m_2 \log m_1 m_2)$.

If we suppose that the sizes of the objects are less than the complexity of the environment we can summarize the total time complexity of this algorithm as $O(n^3 m_1 m_2 m_3 \log m_1 m_2 m_3)$, which improves the algorithm in [Avn89]
($O\left(n^3 m_1 m_2 m_3 \left(\sum_{i \neq j} \log nm_i \log nm_j + (m_1+m_2)(m_1+m_3) \log m_1 m_2 m_3\right)\right)$).

In the case of objects with constant complexity, Avnaim's algorithm which runs in $O(n^3 \log^2 n)$ is improved here in $O(n^3)$.

5 Conclusion

Part 2 establishes some results about the shape of the free-space of polygons moving in translation.

Part 3 is devoted to search double contacts in the case of three objects. In special cases of three convex polygons *or* of a rectangular environment we proved the existence of a double contact. In the general case, an example of non empty free-space without double-contact is given and some results about the complexity and the form of such a free space are shown.

In the case of two or three convex objects in a polygonal environment we deduce, from the existence of double contact, an algorithm to detect a placement without collisions which improves the preceding algorithms.

An interesting problem in this topic is to establish lower bound. Lower bounds exists in the case of the computation of all the solutions by computing the result's size, but in the case of searching only one solution we do not know if our upper bounds are tight or not.

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References

[AB87] F. Avnaim and J.D. Boissonnat. Simultaneous containment of several polygons. In *Third ACM Symposium on Computational Geometry in Waterloo*, June 1987.

[Avn89] F. Avnaim. *Placement et déplacement de formes rigides ou articulées.* Thèse de doctorat en sciences, Université de Franche-Comté, (France), June 1989.

[Ede87] H. Edelsbrunner. *Algorithms on Combinatorial Geometry.* Springer-Verlag, 1987.

[For85] S. J. Fortune. Fast algorithms for polygon containment. In *12th Colloquium on Automata, Languages and Programming*, pages 189–198, Springer-Verlag, 1985.

[GRS83] L.J. Guibas, L. Ramshaw, and J. Stolfi. A kinematic framework for computational geometry. In *IEEE Symposium on Foundations of Computer Science*, pages 100–111, 1983.

[LW79] T. Lozano-Perez and M.A. Wesley. An algorithm for planning collision free paths among polyhedral obstacles. *Communications of the ACM*, 560–570, 1979.